Matemática Aplicada II – TEA013 – UFPR - Prof. Maurício Gobbi

~~Exame Final~~ P1 – 27/03/2026 – Aluno:

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1 [30] Calcule a transformada de Laplace da função

$$f(t) = H(t-5) - H(t-1)$$

Onde $H(t)$ é a função de Heviside. Lembre-se que:

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \int_0^{+\infty} f(t)e^{-st}dt$$

SOLUÇÃO:

Laplace na f: $\bar{f}(s) = \int_0^{\infty} H(t-5)e^{-st}dt - \int_0^{\infty} H(t-1)e^{-st}dt$

$$= \cancel{\int_5^{\infty} e^{-st}dt} - \left[\int_1^{5} e^{-st}dt + \cancel{\int_5^{\infty} e^{-st}dt}\right] = \left.\frac{e^{-st}}{-s}\right|_1^5$$

$$= \frac{e^{-s} - e^{-5s}}{s} \;.//$$

2 [40] Encontre $x(t)$ $(t \geq 0)$ usando transformada de Laplace em: $x'' + 2x' = \delta(t)$. $x(0) = 0,\ \ x'(0) = 2$.

$\mathcal{L}(\delta) = 1$

Lembre-se que:

$$\mathscr{L}\left\{f''(t)\right\} = s^2\bar{f}(s) - f'(0) - s f(0). \quad \mathscr{L}\left\{f'(t)\right\} = s\bar{f}(s) - f(0).$$

$$\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s-a}. \quad \mathscr{L}^{-1}\left\{\bar{f}(s)\bar{g}(s)\right\} = \int_0^t f(t-\tau)g(\tau)d\tau.$$

SOLUÇÃO:

$$s^2\bar{x} - \cancelto{2}{x'(0)} - s\,\cancelto{0}{x(0)} + 2s\bar{x} - \cancelto{0}{x(0)} = 1$$

$$\bar{x} = 3\,\frac{1}{s^2 - s} = 3\,\frac{1}{s}\,\frac{1}{s-1}$$

$$x = \mathcal{L}^{-1}\left\{3\,\frac{1}{s}\,\frac{1}{s-1}\right\} = 3\,\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}}_{1} * \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}}_{e^t} = \int_0^t e^{\tau}d\tau = e^t - 1$$

3 [30] Considere a sequência de funções $\phi_n(x)$ abaixo, onde $n = 1, 2, ..., \infty$. Mostre que $\phi_n(x)$ é uma sequência delta, ou seja, que quando $n \to \infty$, $\phi_n(x) \to \delta(x)$ de Dirac.

$$\phi_n(x) = \begin{cases} 0, \ x < 0 \\ 2n^2 x, \ 0 < x < \frac{1}{n} \\ 0, \ x > \frac{1}{n} \end{cases}$$

$\boxed{y = nx}$

SOLUÇÃO:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \phi_n(x)\,dx = \lim_{n\to\infty} \int_{0}^{1} 2\cancel{n^2}\frac{y}{\cancel{n}}\frac{dy}{\cancel{n}} = \lim_{n\to\infty} \left[y^2 \right]_0^1$$

$$\lim_{n\to\infty} \left[1^2 - 0 \right] = 1$$

$\Longrightarrow$

$y = \frac{x}{n}$, $dy = \frac{dx}{n}$

$x = \frac{1}{n} \Rightarrow y = 1$

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \phi_n(x) f(x)\,dx \overset{\downarrow}{=} \lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n\left(\frac{y}{n}\right) f\left(\frac{y}{n}\right) d\left(\frac{y}{n}\right)$$

$$= \lim_{n\to\infty} \int_0^1 2\cancel{n^2}\frac{y}{\cancel{n}} f\left(\frac{y}{n}\right)\frac{dy}{\cancel{n}} = \lim_{n\to\infty} \int_0^1 2 f\left(\frac{y}{n}\right) y\,dy = 2\int_0^1 f(0)\,y\,dy$$

$$\cancel{2} f(0) \frac{y^2}{\cancel{2}}\Big|_0^1 = f(0)\left(1^2 - 0\right) = f(0) \checkmark \ \underline{\text{ok}}$$