Matemática Aplicada II – TEA013 – UFPR - Prof. Maurício Gobbi

Avaliação 2 – 18/05/2026 – Aluno:

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1 [40] Considere a equação ordinária para $\psi(t)$ $(0 < t < \infty)$:

$$\frac{d^2\psi}{dt^2} + k\frac{d\psi}{dt} = f(t), \quad \psi(0) = \frac{d\psi}{dt}(0) = 0.$$

(a) [10] Escreva o operador diferencial do problema acima.
(b) [20] Encontre a equação diferencial e condições de contorno para a função de Green G associada ao problema acima.
(c) [10] Identifique o operador adjunto e se ele é ou não é auto-adjunto.
(d) [20 EXTRA] Encontre a solução $\psi(t)$ do problema considerando $f(t) = e^{-t}$.

SOLUÇÃO:

(a) $L = \left(\frac{d^2}{dt^2} + k\frac{d}{dt}\right)$.

Integrar por partes 2x

$\delta(\tau - t)$

(b) $\int_0^\infty G\psi'' + kG\psi' \, d\tau = \int_0^\infty G f \, d\tau \Rightarrow G\psi'\Big|_0^\infty - G'\psi\Big|_0^\infty + kG\psi\Big|_0^\infty + \int_0^\infty \psi\underbrace{(G'' - kG')}_{\underbrace{\left(\frac{d^2}{d\tau^2} - k\frac{d}{d\tau}\right)}_{\text{operador adjunto } L^\#}G} d\tau = \int_0^\infty G f \, d\tau$

$$\boxed{\frac{d^2G}{d\tau^2} - k\frac{dG}{d\tau} = \delta(\tau - t), \quad G(\tau=\infty) = \frac{dG}{d\tau}(\tau=\infty) = 0}$$

(c) $L \neq L^\#$ não é auto adjunto

(d) $\frac{d^2G^-}{d\tau^2} - k\frac{d}{d\tau}G^- = 0, \quad \tau < t \Rightarrow$ $\frac{dG^-}{d\tau} = g \Rightarrow \frac{dg}{d\tau} - kg = 0 \Rightarrow g = Ae^{k\tau}$

$dG^- = Ae^{k\tau}d\tau \Rightarrow G^- = A\frac{e^{k\tau}}{k} + B$

Idem para $G^+ \Rightarrow G^+ = C\frac{e^{k\tau}}{k} + D$, $\frac{dG^+}{d\tau}(\tau \to \infty) = 0 = Ce^{k(\infty)} \Rightarrow C = 0$

$G^+(\tau \to \infty) = 0 = D$.

$\int_{t-\epsilon}^{t+\epsilon} d\left(\frac{dG}{d\tau}\right) = \int_{t-\epsilon}^{t+\epsilon} \delta \, d\tau = 1$

$\cancel{\frac{dG^+}{d\tau}}^{\,0}\Big|_{\tau=t} - \frac{dG^-}{d\tau}\Big|_{\tau=t} = 1 \Rightarrow \frac{dG^-}{d\tau}\Big|_{\tau=t} = 1$;

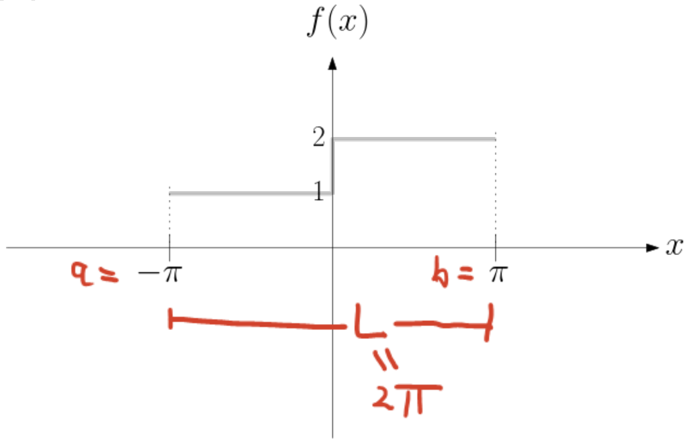
$\Rightarrow G^-(\tau=t) = G^+(\tau=t) = 0$

$\frac{dG^-}{d\tau}(\tau=t) = 1 = Ae^{kt} \Rightarrow A = e^{-kt}$, $G^-(\tau=t) = 0 = B + \frac{e^{k(0)}}{k} \Rightarrow B = -\frac{1}{k}$.

$$\boxed{G = (1 - H(\tau - t))\left(\frac{e^{k(\tau-t)} - 1}{k}\right)} \Rightarrow \psi = \int_0^t \frac{(e^{k(\tau-t)} - 1)}{k} \underbrace{e^{-\tau}}_{f(\tau)} d\tau \Rightarrow$$

$$\Rightarrow \psi(t) = \frac{(1 - e^{-t})}{1 - k} - \frac{1}{k}\frac{(1 - e^{-kt})}{1 - k} \quad //$$

2 [30] Desenvolva a série de Fourier da função mostrada na figura.



$$f(x) = c_0 + \sum_{n=1}^{\infty} A_n \cos\frac{2\pi n x}{L} + B_n \operatorname{sen}\frac{2\pi n x}{L}$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{\cancel{2\pi} n x}{\cancel{L}\ 2\pi} + B_n \operatorname{sen}\frac{\cancel{2\pi} n x}{\cancel{L}\ 2\pi},$$

$$A_n = \frac{2}{L}\int_a^b f(\xi)\cos\frac{2\pi n\xi}{L}\,d\xi, = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\cos(n\xi)\,d\xi$$

$$B_n = \frac{2}{L}\int_a^b f(\xi)\operatorname{sen}\frac{2\pi n\xi}{L}\,d\xi. = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\operatorname{sen}(n\xi)\,d\xi$$

$n=0$ SOLUÇÃO:

$$\frac{A_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\,d\xi = \frac{1}{2\pi}\int_{-\pi}^{0} 1\,d\xi + \frac{1}{2\pi}\int_{0}^{\pi} 2\,d\xi = \frac{1}{2\pi}\left[\xi\Big|_{-\pi}^{0} + 2\xi\Big|_{0}^{\pi}\right] = \frac{1}{2\pi}\left[-(-\pi) + 2\pi\right] = \frac{3}{2}$$

$n \geq 1$:

$$A_n = \frac{1}{\pi}\int_{-\pi}^{0}\cos n\xi\,d\xi + \frac{1}{\pi}\int_{0}^{\pi} 2\cos n\xi\,d\xi = \frac{1}{n\pi}\left(\operatorname{sen} n\xi\right)\Big|_{-\pi}^{0} + \frac{1}{n\pi}\left(2\operatorname{sen} n\xi\right)\Big|_{0}^{\pi} = 0$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{0}\operatorname{sen} n\xi\,d\xi + \frac{1}{\pi}\int_{0}^{\pi} 2\operatorname{sen} n\xi\,d\xi = \frac{1}{n\pi}\left[\left(-\cos n\xi\right)\Big|_{-\pi}^{0} + \left(-2\cos n\xi\right)\Big|_{0}^{\pi}\right] = \frac{1}{n\pi}\left[-1 - 1 + 2 + 2\right] = \frac{2}{n\pi}$$

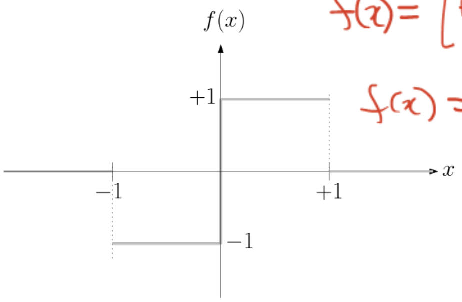
$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\operatorname{sen}(nx)\,dx$$

3 [30] Encontre a Transformada de Fourier da função abaixo definida no eixo real x (ver figura):

$$f(x) = \begin{cases} 0, & \text{se } x < -1 \\ -1, & \text{se } -1 < x < 0 \\ 1, & \text{se } 0 < x < 1 \\ 0, & \text{se } x > 1 \end{cases}$$

$$f(x) = \left[H(x) - H(x-1)\right] + \left[-H(x+1) + H(x)\right]$$

$$f(x) = 2H(x) - H(x-1) - H(x+1)$$



$$\mathscr{F}\{f(x)\}(k) = \widehat{f}(k) \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{-ikx}\,dx.$$

$$f(x) = \int_{-\infty}^{+\infty} \widehat{f}(k)e^{ikx}\,dk.$$

SOLUÇÃO:

$$\hat{f}(k) = \cancelto{0}{\frac{1}{2\pi}\int_{-\infty}^{-1} 0\,dx} + \frac{1}{2\pi}\int_{-1}^{0} -e^{-ikx}\,dx + \frac{1}{2\pi}\int_{0}^{1} e^{-ikx}\,dx + \cancelto{0}{\frac{1}{2\pi}\int_{1}^{\infty} 0\,dx}$$

$$\hat{f}(k) = \frac{1}{2\pi}\left[-\frac{e^{-ikx}}{(-ik)}\Bigg|_{-1}^{0}\right] + \frac{1}{2\pi}\left[\frac{e^{-ikx}}{(-ik)}\Bigg|_{0}^{1}\right]$$

$$\hat{f}(k) = \frac{i}{2\pi k}\left[-1 + e^{ik} + e^{-ik} - 1\right]$$

$$\Downarrow$$

usar

$$\boxed{\cos k = \frac{e^{ik} + e^{-ik}}{2}}$$

$$\boxed{\hat{f}(k) = \frac{2i}{\pi k}(\cos k - 1)}$$